M3031906F1PV1

M303/D

Module Examination 2019

Further Pure Mathematics

Tuesday 11 June 2019 **2.30 pm – 5.30 pm**

Time allowed: 3 hours

The exam is in two parts. Attempt *all* questions in Part 1 (each question is worth 11 marks) and *two* out of the three questions in Part 2 (each question is worth 17 marks). If you submit attempts to all three questions in Part 2, your best two scores will count towards your result.

It should take about two hours to do the Part 1 questions and an hour to do two Part 2 questions.

Start each question on a new page.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles $\circ$ to indicate any 'corner' points on a boundary that are not included and small closed circles $\bullet$ to indicate any 'corner' points that are included.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is *not* permitted in this examination.

NB There was a correction issued to question 6(c)(i).
Instead of saying:

"Show that for $(a, b) \in X, d((1/n, 0), (a, b)) \geq 2|b|$."

The question should have said:

"Show that for $(a, b) \in X, d((1/n, 0), (a, b)) \geq \frac{|a|}{2}$ for $n > \frac{2}{|a|}$."

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PART 1

Attempt all questions. Each question is worth 11 marks.
All of the Part 1 questions count towards your result.

Question 1

(a) Use the Euclidean algorithm to find $\text{hcf}(217, 483)$. [4]

(b) Determine the least positive integer that satisfies all three of the following linear congruences.

$$x \equiv 3 \pmod 4; \quad 3x + 4 \equiv 6 \pmod 7; \quad 7x \equiv 6 \pmod{11}.$$ [5]

(c) Is the following statement true or false? If it is true, prove it. If it is false, justify your answer.

> Any number of the form $17m + 2$, where $m \geq 0$ is a non-negative integer, must have a prime divisor of this same form. [2]

Question 2

(a) Let $G = \mathbb{Z}_9 \times \mathbb{Z}_5$.

(i) Write down the order of G. [1]

(ii) Write down the order of $(3, 0)$. [1]

(iii) Write down an element of order 5. [1]

(iv) Is G cyclic? Justify your answer. [2]

(b) Consider the following two groups, G and H, both of which have order 360.

$$G = \mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$$
$$H = \mathbb{Z}_{30} \times \mathbb{Z}_2 \times \mathbb{Z}_6$$

(i) Write each of G and H as a direct product of cyclic groups of prime power order (and hence deduce that G and H are not isomorphic groups). [2]

(ii) Is either of G or H isomorphic to $\mathbb{Z}_{360}$? Briefly justify your answer. [2]

(iii) Is either of G or H isomorphic to Dic_{90}? Briefly justify your answer. [2]

Question 3

(a) Evaluate the Legendre symbol (127/167). (Note that 127 and 167 are primes.) [3]

(b) Determine whether or not the quadratic congruence

$$3x^2 + 4x + 5 \equiv 0 \pmod{17}$$

has solutions. [3]

(c) Use the rational root test to show that

$$f(x) = x^3 + 12x^2 - 5x + 3$$

is irreducible over $\mathbb{Q}$. [5]

Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 \leq 3\},\ B = \{(a, b) \in \mathbb{R}^2 : a > 0\}$$

and

$$C = A - B.$$

(a) Sketch on separate diagrams, or describe in words, each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Determine whether $(-1, 0)$ is a closure point of C for the Euclidean metric $d^{(2)}$. [2]

(c) Write down in set notation the closure, interior and boundary of C for the Euclidean metric $d^{(2)}$ on the plane. [4]

(d) Determine whether the closure of C for the Euclidean metric on the plane is compact. [2]

Question 5

Let $\alpha = \sqrt{8} - \sqrt{7}$; you may assume that
$[\mathbb{Q}(\sqrt{8}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = 2$, $\sqrt{8} \notin \mathbb{Q}(\sqrt{7})$ and $\sqrt{7} \notin \mathbb{Q}(\sqrt{8})$.

(a) Express α^3 in the form $a\sqrt{8} + b\sqrt{7}$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of (a), or otherwise, show that $\sqrt{7}, \sqrt{8} \in \mathbb{Q}(\alpha)$. [3]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{7}, \sqrt{8})$. [1]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [3]

Question 6

Define a metric, d, on $X = \mathbb{R}^2 - \{(0, b) : b \in \mathbb{R}\}$ by

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + 2|x_2 - y_2|$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.
(You do not have to show that d is a metric.)

(a) Write down $d\big((-1, -1), (1, 1)\big)$. [*1*]

(b) Write down a d-disconnection of X. [*2*]

(c) (i) Show that for $(a, b) \in X$, $d((1/n, 0), (a, b)) \geq 2|b|$. [*2*]

(ii) Hence, or otherwise, show that $(1/n, 0)$ is not a d-convergent sequence in X. [*2*]

(iii) Hence, or otherwise, show that X is not d-complete. [*3*]

(iv) Deduce that X is not d-sequentially compact. [*1*]

PART 2

Attempt two questions. Each question is worth 17 marks. (If you submit attempts to all three questions, your best two scores will count towards your result.)

Question 7

(a) Let D_{50} be the dihedral group of order 100:

$$D_{50} = \langle r, s | r^{50} = s^2 = e, sr = r^{49}s \rangle$$

(i) Show that $r^{25}sr^{25}s = e$. [2]

(ii) Hence (or otherwise) deduce that $r^{25} \in Z(D_{50})$ where $Z(D_{50})$ is the centre of D_{50}. [2]

(b) Let G be an arbitrary group of order $100 = 4 \times 25$.

(i) Write down the order of the Sylow 2-subgroups and the order of the Sylow 5-subgroups of G. [1]

(ii) Calculate the possible numbers of Sylow 2-subgroups of G. [2]

(iii) Prove that there is only one Sylow 5-subgroup of G and hence deduce that this subgroup is normal. [2]

(c) Now let G be a group with order p^2q^2 where p and q are distinct, odd primes, with $p > q$. Let m_p be the number of Sylow p-subgroups and m_q be the number of Sylow q-subgroups.

(i) Show that if $m_p \neq 1$ then p divides $q^2 - 1$. [4]

(ii) Hence (or otherwise) deduce that G has a unique Sylow p-subgroup. [3]

(iii) Hence deduce that G cannot be simple. [1]

Question 8

Let $R = \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} : a, b \in \mathbb{Z}\}$, and let

$$N(a + b\sqrt{-6}) = a^2 + 6b^2$$

be a norm on R, which you may assume is multiplicative.

(a) Show that the only elements $r \in R$ with $N(r) = 1$ are $r = \pm 1$, and that there are no elements with $N(r) = 2$ or $N(r) = 3$. [4]

(b) Show that each of 2, 3 and $\sqrt{-6}$ is irreducible in R. [5]

(c) Hence, or otherwise, show that R is not a unique factorisation domain. [1]

Although the ring $R = \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} : a, b \in \mathbb{Z}\}$ is not a unique factorisation domain, some reducible elements of R may still be uniquely factorisable. Indeed, show that 7 has a unique factorisation in R as follows. Suppose that $7 = uv$ where $u = a + b\sqrt{-6}$, $v = c + d\sqrt{-6}$ and u, v are not units of R.

(d) By taking norms in the equation $7 = uv$ show that the norms of both u and v must be equal to 7. [1]

(e) Show that there are exactly four elements $u = a + b\sqrt{-6}$ in R such that $N(u) = 7$. [2]

(f) Hence show that 7 has a unique factorisation in R. [2]

(g) Using the factorisation in (f) or otherwise, show that the ideal $I = \langle 2, 1 + \sqrt{-6} \rangle$ of R is equal to R, that is, $I = R$.

[Hint: Establish that $1 \in I$.] [2]

Question 9

(a) Let

$$A = \{f \in C[0,1] : |f(x) - f(y)| \leq \exp(|x-y|)|x-y| \text{ for each } x, y \in [0,1]\}$$

and

$$B = \{f \in C[0,1] : |f(x) - 2019| \leq 1 \text{ for } 0 \leq x \leq 1\}.$$

(i) Show that A is $d_{\max}$-closed. [3]

(ii) Deduce that $A \cap B$ is $d_{\max}$-closed. [1]

(iii) Hence or otherwise show that $A \cap B$ is $d_{\max}$-sequentially compact. (You may assume that A is an equicontinuous set of functions.) [2]

(b) Define a distance function $d\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ as follows:

$$d(a,b) = \begin{cases} 0, & \text{if } a = b = 0, \\ 1 + \max\{a,b\}, & \text{if } \min\{a,b\} = 0 \text{ and } \max\{a,b\} > 0 \\ |a-b|, & \text{if } \min\{a,b\} > 0. \end{cases}$$

(i) Write down $d(0,1)$. [1]

(ii) Prove that d is a metric on $[0,\infty)$ by showing that

(1) d satisfies (M1). [2]

(2) d satisfies (M2). [2]

(3) d satisfies (M3). [6]

[END OF QUESTION PAPER]